Applied Physics 195 / Physics 195 — Assignment #2

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Due: **5pm SHARP (4:50pm + 10 min grace period)**, October 2nd, 2015; slide your work under through the door at Maxwell-Dworkin Room 131.

Problem 1 (60 pt): Kinetic inductance of free electron Fermi gas

Consider a 3D conductor with a cross sectional area A, a length l, and a total of N free electrons. The electron number density per unit volume is then $n_{0,3D} = N/(Al)$. Let m be the electron mass. In class we used the Drude equation of motion to derive the kinetic inductance L_K along the length direction (to which we assign x-axis):

$$L_K = \frac{l}{A} \times \frac{m}{n_{0,3D}e^2}. \tag{1}$$

Here we re-derive L_K from the Fermi gas point of view. In thermal equilibrium at temperature T, electrons are distributed in the Fermi space ($\vec{k}$-space) according to the Fermi-Dirac statistics $f(\epsilon(\vec{k}))$, where

$$\epsilon(\vec{k}) = \frac{\hbar^2 k^2}{2m} \tag{2}$$

is the free electron energy. As we do *not* assume $T = 0$, this Fermi space electron distribution does not coincide with the Fermi sphere in general. An electric field is then applied along the conductor length starting at $t = 0$. This will accelerate each electron along the k_x direction in the Fermi space. Let $\delta\hat{k_x}$ be the resulting displacement of each electron (and thus of the entire electron distribution) in the Fermi space at time $t = \Delta t$ ($\hat{k_x}$: unit vector in the $+k_x$ direction).

(a) Argue that the increase in the total energy of the electron distribution as it shifts from $t = 0$ and $t = \Delta t$—the increase amount is the collective kinetic energy E_K of the electron gas—is given by

$$E_K = g \times \frac{Al}{(2\pi)^3} \int \epsilon(\vec{k}) f(\epsilon(\vec{k} - \delta\hat{k_x})) d^3\vec{k} - g \times \frac{Al}{(2\pi)^3} \int \epsilon(\vec{k}) f(\epsilon(\vec{k})) d^3\vec{k}, \tag{3}$$

where $g = 2$ accounts for the spin degrees of freedom and each integration is taken over the entire Fermi space. Carry out this calculation to the leading non-vanishing order of δ.

(b) Argue that the current I due to the movement of the electrons is given, at time $t = \Delta t$, by

$$I \quad = \quad g \times \frac{A}{(2\pi)^3} \int e v_x(\vec{k}) f(\epsilon(\vec{k} - \delta\hat{k_x})) d^3\vec{k} \tag{4}$$

where $g = 2$ (spin degrees of freedom), $v_x(\vec{k}) = (1/\hbar)(\partial\epsilon/\partial k_x) = \hbar k_x/m$ is the single electron group velocity, and the integration is all across the Fermi space. Carry out this calculation (no approximation is needed).

(c) From the results of (a) and (b), you will see $E_K \sim I^2$. From this, evaluate L_K, and show its agreement with Eq. (1).

(d) For a 2D free electron Fermi gas,

$$L_K = \frac{l}{W} \times \frac{m}{n_{0,2D}e^2} \tag{5}$$

where $n_{0,2D}$ is the electron number density per unit area and W is the cross sectional width of the 2D conductor. Prove this first by using the Drude equation of motion in 2D, and second by adapting the Fermi gas formalism above to the 2D case.

Problem 2 (60 pt): Kinetic inductance with general $\epsilon(\vec{k})$, Drude formalism, & graphene example

(a) We now expand the formalism of Problem 1 to the case of general electron energy dispersion $\epsilon(\vec{k})$, which is not any more the free electron energy dispersion of Eq. (2), due to the Coulomb interaction of the electron with the background periodic lattice of positive ions.[1] Eqs. (3) and (4) still hold for the general $\epsilon(\vec{k})$ and $v_x(\vec{k}) = (1/\hbar)(\partial\epsilon/\partial k_x)$ for 3D conductor. By calculating Eqs. (3) and (4) to non-vanishing leading orders of δ (feel free to assume $\epsilon(\vec{k}) = \epsilon(k)$ for brevity), show

$$L_K = \frac{l}{A} \times \frac{\hbar^2}{ge^2} \left[\int \frac{d^3\vec{k}}{(2\pi)^3} \frac{\partial^2\epsilon}{\partial k_x^2} f(\epsilon(\vec{k})) \right]^{-1} \tag{6}$$

where the integration is all across the Fermi space (to be precise, in the presence of the interaction of electrons with the background periodic lattice of positive ions, which yield the general $\epsilon(\vec{k})$, the integration actually should be over what we will study later as a conduction band, but let's not worry about this for now).

(b) For a 2D electron gas with general $\epsilon(\vec{k})$ (once again feel free to assume $\epsilon(\vec{k}) = \epsilon(k)$), show

$$L_K = \frac{l}{W} \times \frac{\hbar^2}{ge^2} \left[\int \frac{d^2\vec{k}}{(2\pi)^2} \frac{\partial^2\epsilon}{\partial k_x^2} f(\epsilon(\vec{k})) \right]^{-1}. \tag{7}$$

(c) By applying Eq. (7) to graphene (an example 2D conductor) with[2] $\epsilon(\vec{k}) = \hbar v_F k$ (in graphene $v_F \sim 10^6$ m/s is a constant independent of electron density) and by assuming $T \ll T_F$ and thus $T \approx 0$, prove

$$L_K = \frac{l}{W} \times \sqrt{\frac{4\pi}{g}} \times \frac{\hbar}{v_F e^2} \times \frac{1}{\sqrt{n_{0,2D}}}. \tag{8}$$

For this, you will have to prove and use $k_F^2 = 4\pi n_{0,2D}/g$ in 2D. Now in graphene, g is 4, not 2, due to the valley degrees of freedom (we will study this later) in addition to the spin degrees of freedom. With $g = 4$,

$$L_K = \frac{l}{W} \times \frac{\sqrt{\pi}\hbar}{e^2 v_F} \times \frac{1}{\sqrt{n_{0,2D}}}. \tag{9}$$

$L_K \sim 1/\sqrt{n_{0,2D}}$ for graphene with $\epsilon(\vec{k}) \sim k$ contrasts $L_K \sim 1/n_{0,2D}$ of the 2D free electron gas [Eq. (5)] where $\epsilon(\vec{k}) \sim k^2$ with the well defined electron mass m.

(d) In analogy to Eq. (5), we can define an analogous mass m_p for a graphene electron by equating Eq. (9) to $l/W \times m_p/(n_{0,2D}e^2)$. Show that $m_p = \epsilon_F/v_F^2$. This mass m_p, originating from the collective electron acceleration associated with the kinetic inductance, is the effective collective dynamical mass of graphene electrons normalized to the number of electrons, or *plasmonic mass*. If you wish to use the Drude equation of motion for graphene electrons, you should use this plasmonic mass in conjunction with Newton's 2nd law.

(e) More generally, for an arbitrary $\epsilon(\vec{k})$, show that the plasmonic mass, which should be used in the Drude equation of motion, is given by

$$m_p = \frac{n_0\hbar^2}{g} \left[\int \frac{d^D\vec{k}}{(2\pi)^D} \frac{\partial^2\epsilon}{\partial k_x^2} f(\epsilon(\vec{k})) \right]^{-1} \tag{10}$$

where D is the conductor dimension (2 or 3), $n_0 = n_{0,2D}$ for $D = 2$, and $n_0 = n_{0,3D}$ for $D = 3$.

[1] Calculation of such general $\epsilon(\vec{k})$ will be an important focus of this course soon.
[2] We will derive this in a future lecture.

Problem 3 (60 pt; NO COLLABORATION ALLOWED): Lattice waves in 1D diatomic crystal

Consider a 1D diatomic crystal as shown on page 6 of Lecture #4. Assume $C_1 = C_2 \equiv C$ and $M_1 \neq M_2$ (these parameters are with reference to the said page of Lecture #4). The entire crystal length is L, the length of the unit cell (containing two atoms) is a, the total number of unit cells is $L/a = N$, and the total number of atoms is $2N$.

(a) Derive all possible lattice-wave dispersion relations and sketch them in the 1st Brillouin zone.

(b) How many available k states are there? What is the total number of normal modes? What are the total degrees of freedom of the system?

Problem 4 (60 pt): Lattice waves in 2D monatomic crystal

Consider a 2D square monatomic crystal with a lattice constant of a. In equilibrium, atoms (single atomic mass: M) are located at (na, ma) where $n, m = 1, 2, 3, ..., N$. Thus there are a total of N^2 atoms and a total of N^2 unit cells. Let the atom at equilibrium position (na, ma) be displaced by $(x_{n,m}, y_{n,m})$. In the harmonic approximation and considering only the nearest neighborhood interactions, we can write the total potential energy of the crystal as

$$V = \sum_{n,m} \left[\frac{1}{2} C_1 (x_{n,m} - x_{n+1,m})^2 + \frac{1}{2} C_2 (y_{n,m} - y_{n+1,m})^2 + \frac{1}{2} C_1 (y_{n,m} - y_{n,m+1})^2 + \frac{1}{2} C_2 (x_{n,m} - x_{n,m+1})^2 \right]. \tag{11}$$

The 1st term means that if any two atoms adjacent along the x axis are perturbed along the x axis, a harmonic restoring force acts along the x axis. The 2nd term means that if any two atoms adjacent along the x axis are perturbed along the y axis, a harmonic restoring force acts along the y axis. The former and latter harmonic forces differ in magnitude for the same degree of perturbation, and hence, the two different coefficients, C_1 and C_2. Furthermore, it is reasonable to assume that $C_1 > C_2$; for numerical concreteness, let's assume that C_2 is 10 percent of C_1. The 3rd and 4th terms can be similarly interpreted for any two atoms adjacent along the y axis.

(a) Find all possible 2D lattice wave dispersion relations $\omega(\vec{k})$.

(b) Sketch, within the 1st Brillouin zone, these dispersion relations for the lattice waves propagating along the x direction.

(c) Evaluate the speed of sound as a function of direction for every possible branch of dispersion relations. Along which direction does the sound propagate fastest?

(d) How many available k states are there? What is the total number of normal modes? What are the total degrees of freedom of the system?