Physics 195 / Applied Physics 195 — Assignment #9

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Due: **5pm SHARP (4:50pm + 10 min grace period)**, December 4th, 2015 @ MD Rm 131.

Problem 1 (100 pt): Carrier concentration and transport in semiconductor

(a) Consider a semiconductor with valence band maximum ϵ_v and conduction band minimum ϵ_c (band gap: $\epsilon_g = \epsilon_c - \epsilon_v$). Both conduction and valence bands may be approximated with quadratic dispersion with density-of-states effective mass of m_e^* and m_h^*, which would take into account any possible quadratic dispersion anisotropy and valley degeneracy. Assume $\epsilon_v < \mu < \epsilon_c$, $\epsilon_c - \mu \gg k_B T$, and $\mu - \epsilon_v \gg k_B T$ for chemical potential μ. Show that conduction band electron density n and valence band hole density p are

$$n = 2\left(\frac{m_e^* k_B T}{2\pi\hbar^2}\right)^{\frac{3}{2}} \exp\left(-\frac{\epsilon_c - \mu}{k_B T}\right); \tag{1}$$

$$p = 2\left(\frac{m_h^* k_B T}{2\pi\hbar^2}\right)^{\frac{3}{2}} \exp\left(\frac{\epsilon_v - \mu}{k_B T}\right). \tag{2}$$

Show that np is independent of μ (law of mass action): regardless of doping, np stays constant; if n-doped, $n \uparrow$ and $p \downarrow$ and if p-doped, $n \downarrow$ and $p \uparrow$, but np is the same. Prove that intrinsic chemical potential μ_i is

$$\mu_i = \epsilon_c - \frac{1}{2}\epsilon_g + \frac{3}{4} k_B T \ln\left(\frac{m_h^*}{m_e^*}\right) \tag{3}$$

and thus is *almost* at the middle of the band gap. Let n_i and p_i be intrinsic n and p ($n_i = p_i$). Show

$$n = n_i \exp\left(\frac{\mu - \mu_i}{k_B T}\right); \tag{4}$$

$$p = p_i \exp\left(\frac{\mu_i - \mu}{k_B T}\right). \tag{5}$$

Argue that if the semiconductor is n-doped, $\mu > \mu_i$ and if p-doped, $\mu < \mu_i$.

(b) Calculate $n_i = p_i$ and μ_i for silicon at 300K. Use $m_e^* = 1.1 m_0$ and $m_h^* = 0.55 m_0$. The m_h^* value given here is not most accurate: as we discussed in Lecture #18, the behavior of the 3 silicon valence bands around their maxima at the Γ point of the 1st Brillouin zone are slightly complicated; while this can be modeled more accurately, let's just use the m_h^* value given above, as it will not prevent us from seeing the essence.

(c) A silicon sample is doped with 10^{16} cm^{-3} boron atoms and a certain number of donors. μ is 0.36 eV above μ_i at 300K. What is the donor concentration N_d?

(d) A silicon sample is doped with 10^{17} cm^{-3} arsenide atoms. Calculate n, p, and μ at 300 K. Repeat the problem for a silicon sample doped with 10^{17} cm^{-3} boron atoms.

(e) Estimate arsenic donor density N_d required to make the conductivity 10^4 times greater than the intrinsic conductivity of a crystalline silicon at 300 K. Use[1] $\mu_e = 450$ cm^2/V·s and $\mu_h = 150$ cm^2/V·s for electron and hole mobility, and assume—rather unrealistically—that these mobilities stay the same after the doping.

(f) Show that the Hall coefficient R_H of a semiconductor is given by

$$R_H = \frac{1}{e}\frac{p\mu_h^2 - n\mu_e^2}{(p\mu_h + n\mu_e)^2}. \tag{6}$$

Once again, μ_e and μ_h refer to electron and hole mobilities, not chemical potentials (see footnote 1).

[1] μ is a common notation for both chemical potential and mobility; in this set μ is mobility if subscripts 'e' or 'h' accompany.

Problem 2 (100 pt; no collaboration): Carrier concentration and chemical potential in semimetal

A semimetal has an energy overlap of $\epsilon_o > 0$ between valence and conduction bands. That is, the valence band maximum ϵ_v and the conduction band minimum ϵ_c are related by $\epsilon_v - \epsilon_c = \epsilon_o$. The conduction band has g equivalent minimum valleys, with each valley having an ellipsoidal quadratic dispersion with longitudinal effective mass of m_L^* and transverse effective mass of m_T^*. The valence band has no valley degeneracy and has a spherical quadratic dispersion with an effective mass of m_S^*. Determine the position of the chemical potential μ—or Fermi level, ϵ_F; that is, you may assume $T_F \gg T$—, the conduction band electron concentration, and the valence band hole concentration in terms of the given parameters.

Problem 3 (80 pt): Semiconductor junctions

(a) A silicon pn junction has $N_a = 10^{17}$ cm^{-3} on the p side, and $N_d = 10^{16}$ cm^{-3} on the n side. At 300K, calculate the chemical potential μ for each region, and draw the equilibrium band diagram. Find the built-in potential V_0 from the diagram. Succinctly explain the transport mechanism of the majority charge carriers (electrons from the n-doped region and holes from the p-doped region) when the pn junction is forward biased. What do the majority charge carriers do if the junction is reverse biased? (I am not asking you about minority charge carriers, as they only give rise to a very small leakage current).

(b) Consider a p^+np bipolar junction transistor with $p+$ region called emitter, n region base, and p region collector. Draw the equilibrium energy band diagram. Also draw the band diagram when the p^+n junction is forward biased and the np junction reverse biased. Under this bias arrangement (called forward active bias), explain the transport of the majority charge carriers (holes from the p^+ and p regions, and electrons from the n region), drawing what is similar to the top figure on page 5 of Lecture #21. The key device point is that the current between the collector and emitter terminals can be controlled by using the voltage of the 3rd (base) terminal, while not drawing much current into the 3rd terminal; can you explain this mechanism in conjunction with your transport picture of majority charge carriers?

Problem 4 (150 pt; no collaboration): Plasmonic excitation in 3D conductor

This problem seeks to guide you through the concept and technical details of Lecture #22, whose big picture I outlined in class. The plasmonic excitation in 3D conductor can be described by solving for the local electron density $n(\vec{r},t) = n_0 + \delta_n(\vec{r},t)$ and local electron velocity $\vec{v}(\vec{r},t)$, where n_0 is the equilibrium electron density (which is charge balanced by the same density of background positive ions). Key dynamical equations are:

$$m_p \frac{d\vec{v}}{dt} = -e\vec{E} - \frac{1}{n}\vec{\nabla}P; \tag{7}$$

$$\frac{\partial n}{\partial t} = -\vec{\nabla}\cdot(n\vec{v}). \tag{8}$$

Eq. (8) states the conservation of overall electron number (continuity equation). Eq. (7) is the equation of motion. The left hand side of Eq. (7) represents the inertial acceleration of the plasmonic mass m_p (collective mass per electron). Its right hand side consists of Coulomb and Pauli restoring forces: the Coulomb restoring force $-e\vec{E}$ arises due to the electron density perturbation (see Eq. (11)); the Pauli restoring force also arises due to the electron density perturbation, as it causes the gradient in electron degeneracy pressure P. Using $\vec{\nabla}P = (\partial P/\partial n)_{n_0}\vec{\nabla}n$ and assuming that δ_n and $\vec{v}$ are small perturbations from equilibrium, show that Eqs. (7) and (8) may be linearized to

$$\frac{\partial \vec{v}}{\partial t} = -\frac{e}{m_p}\vec{E} - \frac{\alpha}{n_0}\vec{\nabla}\delta_n; \tag{9}$$

$$\frac{\partial \delta_n}{\partial t} = -n_0\vec{\nabla}\cdot\vec{v}, \tag{10}$$

where $\alpha \equiv (1/m_p)\,(\partial P/\partial n)_{n_0}$. In addition to these two dynamical equations, you need Maxwell's equations:

$$\vec{\nabla}\cdot\vec{E} = -\frac{e\delta_n}{\epsilon_0} \tag{11}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{12}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{13}$$

$$\vec{\nabla} \times \vec{B} = -\mu_0 n_0 e \vec{v} + \epsilon_0 \mu_0 \frac{\partial E}{\partial t} \tag{14}$$

For example, Eq. (11) is a vital self-consistency statement that connects the local electron density perturbation δ_n to the local Coulomb restoring field $\vec{E}$, which is used in Eq. (9). Take another example with Eq. (14): this connects the local Coulomb restoring field $\vec{E}$ with the local electron velocity $\vec{v}$ or current[2] $-n_0 e\vec{v}$. Generally put, the Maxwell's equations can be used to determine the electric and magnetic fields generated by, and also affecting, the collective dynamics of electrons.

(a) **Longitudinal plasmonic mode**

We first solve for a longitudinal mode, where $\vec{E}$ is parallel to the wave vector $\vec{k}$, *i.e.*, $\vec{\nabla} \times \vec{E} = 0$.

- Under $\vec{\nabla} \times \vec{E} = 0$ ($\vec{k} \parallel \vec{E}$), show $\vec{v} \parallel \vec{E} \parallel \vec{\nabla}\delta_n$; that is, the electron density, electron velocity, and electric field all locally vibrate along the wave propagation direction. Notice that $\vec{B} = 0$ under $\vec{\nabla} \times \vec{E} = 0$, which means that this longitudinal mode cannot give rise to an electromagnetic radiation.
- Under $\vec{\nabla} \times \vec{E} = 0$, show that δ_n and $\vec{v}$ are described by the following wave equations:

$$\frac{\partial^2 \delta_n}{\partial t^2} + \omega_p^2 \delta_n - \alpha \nabla^2 \delta_n = 0; \tag{15}$$

$$\frac{\partial^2 \vec{v}}{\partial t^2} + \omega_p^2 \vec{v} - \alpha \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) = 0. \tag{16}$$

  Using harmonic analysis, show that the dispersion relation of this longitudinal mode is given by

$$\omega^2 = \omega_p^2 + \alpha k^2 \tag{17}$$

  where $\omega_p = (n_0 e^2/m_p \epsilon_0)^{1/2}$ is the plasma frequency. This longitudinal mode is the plasmonic wave where the electron density is spatiotemporally modulated according to Eq. (15).

- The formalism above applies to not only conductors with dense population of electrons, but also classical plasma with rarefied electrons (*e.g.*, ionosphere, laboratory glow discharge, solar wind, interstellar medium, intergalactic medium, and many more). In classical plasma: $m_p = m_0$; n_0 is typically small and yields a low plasma frequency ω_p; and α arises not from electron degeneracy pressure but from thermal pressure. The longitudinal mode for classical plasma is called Langmuir wave, and the longitudinal dispersion of Eq. (17) is called Bohm-Gross dispersion for classical plasma. In classical plasma, α due to thermal pressure is proportional to temperature: see Part (e). When α cannot be ignored at high enough temperature, the classical plasma is called 'warm.' If α is ignorable at low enough temperature, the classical plasma is 'cold.' In conductors, α due to electron degeneracy pressure is not temperature dependent to the first order: see Part (e).
- If we ignore the Pauli restoring force (set α to 0), Eqs. (15) and (16) collapse to time differential equations with no spatial derivatives:

$$\frac{\partial^2 \delta_n}{\partial t^2} + \omega_p^2 \delta_n = 0; \tag{18}$$

$$\frac{\partial^2 \vec{v}}{\partial t^2} + \omega_p^2 \vec{v} = 0. \tag{19}$$

  These correspond to the longitudinal bulk oscillation at a single frequency ω_p discussed in Section B of Lecture #22. This bulk oscillation is not a wave (*i.e.*, physical entities are not spatially modulated),

[2]This current that appears in the first term on the right hand side of Eq. (14) should be rigorously $-ne\vec{v}$ not $-n_0 e\vec{v}$, but we take the linear approximation ignoring the product of δ_n and $\vec{v}$.

or it is a wave with infinite wavelength. Thus you can appreciate that the Pauli restoring force is vital to create the longitudinal plasmonic wave in a 3D conductor while the Coulomb restoring force alone only creates the bulk oscillation (as we will discuss in Lecture #23, this is not the case with 2D conductors, where only Coulomb restoring force without Pauli restoring force can create plasmonic waves). Calculate the bulk oscillation frequency $\omega_p/2\pi$ for gold, silver, and copper. In what spectral regime do these frequencies fall? We have so far ignored electron scattering. If many plasmonic oscillation cycles occur within the electron scattering time τ, that is, if $\omega_p\tau \gg 1$, we can readily observe the bulk plasma oscillation (as well as the longitudinal plasmonic mode, as it can occur only for $\omega > \omega_p$; see Eq. (17)). Calculate $\omega_p\tau$ for gold, silver, and copper.

(b) **Transverse electromagnetic mode**

We now solve for a transverse mode, where $\vec{E} \perp \vec{k}$, that is, $\vec{\nabla} \cdot \vec{E} = 0$.

- Under $\vec{\nabla} \cdot \vec{E} = 0$ ($\vec{k} \perp \vec{E}$), show $\vec{k} \perp \vec{B}$ and $\vec{E} \perp \vec{B}$ while $\vec{E} \parallel \vec{v}$ and $\delta_n = 0$. This is a transverse electromagnetic wave, whose electric field grabs and moves electrons perpendicularly to the wave propagation direction. Spatiotemporal electron density modulation does not occur. Argue that an electromagnetic wave irradiated onto 3D conductor will propagate through the conductor—if a certain condition for the frequency is met; see below—in this transverse mode instead of the longitudinal mode.
- Under $\vec{\nabla} \cdot \vec{E} = 0$, show that the local electron velocity $\vec{v}$ is described by the following wave equation:

$$\frac{\partial^2 \vec{v}}{\partial t^2} + \omega_p^2 \vec{v} - c^2 \nabla^2 \vec{v} \quad = \quad 0. \tag{20}$$

  Using harmonic analysis, show that the dispersion relation for this transverse mode is given by

$$\omega^2 = \omega_p^2 + c^2 k^2. \tag{21}$$

  This transverse mode is the electromagnetic mode that propagates through the conductor, if the incoming electromagnetic wave satisfies $\omega > \omega_p$ ('ultraviolet transparency' for metals). If $\omega < \omega_p$ for which k becomes imaginary, the electromagnetic wave radiated onto the conductor is reflected. This formalism applies again also to classical plasma; reflecting a radio wave off of the ionosphere with $\omega < \omega_p$ is a well-known example.

(c) **General approach that subsumes the results of Parts (a) and (b)**

We can study the collective electron excitation without assuming any particular mode (*e.g.*, longitudinal or transverse) of excitation *a priori*. By applying harmonic analysis to Eqs. (9-14) generally, show that

$$(\omega^2 - \omega_p^2 - \alpha k^2)\vec{E}_\parallel + (\omega^2 - \omega_p^2 - c^2 k^2)\vec{E}_\perp = 0 \tag{22}$$

where $\vec{E} = \vec{E}_\parallel + \vec{E}_\perp$, $\vec{k} \parallel \vec{E}_\parallel$, and $\vec{k} \perp \vec{E}_\perp$. From this general secular equation, argue that Eq. (17) and (21) naturally emerge as dispersion relations for the longitudinal and transverse mode.

(d) **Wave propagation speed and dielectric constant**

Calculate the wave propagation velocity (phase velocity) v_L and v_T—not to be confused with the local electron velocity $\vec{v}$—for the longitudinal plasmonic wave and the transverse electromagnetic wave as functions of ω. From this, determine the frequency-dependent dielectric constants for both modes.

(e) **The α factor**

Earlier α was defined as

$$\alpha = \frac{1}{m_p}\left(\frac{\partial P}{\partial n}\right)_{n_0}. \tag{23}$$

As seen above, α affects only the longitudinal plasmonic mode. Here we seek to calculate α for both classical plasma and conductor.

- We first consider classical plasma where α arises from thermal pressure. As the longitudinal plasmonic mode happens too fast to allow significant heat to flow, use the adiabatic relation $P \propto n^{\gamma}$ to show $\alpha = (\gamma k_B T/m_0)$, where γ is the ratio of the specific heat at constant pressure to that at constant volume. Recall from the ideal gas study that γ is related to the degree of freedom of electron f in the following manner: $\gamma(f) = (f+2)/f$. If $f = 3$ is used, $\gamma(3) = 5/3$ and $\alpha = 5/3 \cdot (k_B T/m_0)$. However, the plasmonic wave is a high-frequency phenomenon where the pressure gradient parallel to $\vec{k}$ is only relevant. Thus, $f = 1$, $\gamma = 3$, and $\alpha = 3 \cdot (k_B T/m_0)$ would be more correct. The correction factor to make transition from 'static' α to 'dynamic' (and correct) α is $\gamma(1)/\gamma(3) = 9/5$.

- For conductor, assuming quadratic dispersion with an effective mass m^* for an individual electron, show that α arising from the standard static electron degeneracy pressure is given by $\alpha = (1/3)v_F^2$. As in classical plasma, this static α is not most correct, for the plasma excitation is a high frequency phenomenon where one-directional dynamic pressure is only relevant. The correction factor is operationally the same as the classical plasma—9/5—(this can be justified from what is called random phase approximation (RPA)), and $\alpha = (3/5)v_F^2$ is a more correct result.