Applied Physics 216 — Assignment #1

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Due: **10:20am + 10 min grace period**, Feb 10th, 2017; slide your work under through the door at Maxwell-Dworkin Room 131.

Problem 1 (100 pt): Coherent excitation of a 2-level atom (example via magnetic resonance)

Consider a ^{1}H proton spin 1/2 with gyromagnetic ratio γ subjected to a static magnetic field $\vec{B_0} = B_0\hat{z}$. At $t = 0$, the proton is prepared in the spin down state $|\psi(t = 0)\rangle = |-\rangle$. At $t = 0$, we apply a time-varying magnetic field $\vec{B_1}(t)$ at a frequency $\omega = \omega_0 = \gamma B_0$ along the y-axis:

$$\vec{B_1}(t) = -2B_1 \cos(\omega t)\hat{y}. \tag{1}$$

Note the negative sign to the field expression above. Assume $B_0 \gg B_1$. Assume no relaxation whatsoever.

(a) The spin state at time t can be written as $|\psi(t)\rangle = c_+(t)|+\rangle + c_-(t)|-\rangle$. Solve the time-dependent Schrödinger equation to calculate $c_+(t)$ and $c_-(t)$ including all the relevant phases of the coefficients. Calculate $P_+(t)$ and $P_-(t)$, the time-varying probabilities for the proton to be in the $|+\rangle$ and $|-\rangle$ states, respectively. Use the rotating wave approximation but for this first time, you should delineate the process to eliminate the fast-varying terms.

(b) Using $|\psi(t)\rangle$ obtained in part (a), calculate $\langle\vec{S}\rangle = \langle S_x\rangle\hat{x} + \langle S_y\rangle\hat{y} + \langle S_z\rangle\hat{z}$, the expected value for the spin angular momentum operator $\vec{S}$, and interpret the motion described by $\langle\vec{S}\rangle$, telling apart the slow Rabi oscillation at frequency $\omega_1 = \gamma B_1$ and the fast Larmor precession at frequency ω_0.

(c) At $t = 0$, upon the application of the time-varying magnetic field $\vec{B_1}(t)$, in what transverse direction is $\langle\vec{S}\rangle$ tipped away from the z-axis? (Mathematically, this is related to the phases of the c_+ and c_- coefficients).

(d) At $t = 3\pi/(2\omega_1)$, evaluate the exact direction $\langle\vec{S}\rangle$ is pointing.

(e) Consider a general excitation frequency ω, which is not any more ω_0 in general. Calculate $P_+(t)$ and $P_-(t)$ (following the steps of Lecture #1, you should show the entire procedure). Plot the amplitude of $P_+(t)$ as a function of ω. Estimate the 3-dB excitation bandwidth.

(f) Set ω back to ω_0. By solving the classical equation of motion

$$\frac{d\vec{S}}{dt} = \gamma\vec{S} \times \left[\vec{B_0} + \vec{B_1}(t)\right] \tag{2}$$

where $\vec{S}$ here is not any more an operator but a classical angular momentum, calculate $\vec{S}(t)$ and check its consistency with the result you obtained from part (b).

Problem 2 (30 pt): Equilibrium magnetization

An ensemble of N isolated protons per unit volume subjected to a static magnetic field B_0 are in thermal equilibrium at 300 K. Calculate B_0 at which the population density difference between $|+\rangle$ and $|-\rangle$, normalized to N, is 6×10^{-5} (that is, $(N_{|+\rangle} - N_{|-\rangle})/N = 6 \times 10^{-5}$). At this B_0 calculate the Larmor frequency ω_0. Repeat the problem for an ensemble of N isolated electrons.

Problem 3 (100pt): Excitation of a collection of 2-level atoms in the presence of T_1 and T_2 relaxations (example via magnetic resonance)

Consider now a collection of N (per unit volume) ^{1}H proton spins in a static magnetic field $\vec{B_0} = B_0\hat{z}$. We denote their equilibrium magnetization M_0. At $t = 0$ (in thermal equilibrium), $\vec{M}(t = 0) = M_0\hat{z}$. Let the spin-spin and spin-lattice relaxation times be T_2 and T_1. At $t = 0$, we apply a time-varying magnetic field $\vec{B_1}$ at the Larmor frequency $\omega = \omega_0$ along the y-axis:

$$\vec{B_1} = 2B_1 \cos(\omega_0 t)\hat{y}. \tag{3}$$

Note that differently from Problem 1, there is no negative sign in the expression above. Assume $B_0 \gg B_1$.

(a) Following the steps of Lecture #3, convert the Bloch equation into the differential equations for $T(t)$, the slowly varying factor of the transverse magnetization component, $M_x\hat{x} + M_y\hat{y}$, and for $L(t) = M_z(t)$.

(b) Solve, explicitly, for $T(t)$ and $L(t)$, assuming the large-signal (coherent) regime behavior, that is, $\omega_1 T_2 \gg 1$ and express the saturation magnetization, $L_{sat} = M_{z,sat} = \lim_{t\to\infty} L(t)$, in terms of M_0, ω_1, T_1, and T_2. Express the characteristic time to reach the saturation.

(c) Evaluate, numerically, $M_{z,sat}/M_0$ for $T_2 = 1$ s, $T_1 = 3$ s, and $B_1 = 10^{-3}$ T. If $B_0 = 10$ T and temperature is 300 K, what is the normalized population difference $(N_{|+\rangle} - N_{|-\rangle})/N$ for $t = 0$ and that for $t = \infty$?