Applied Physics 216 — Assignment #2

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Due: **10:20am + 10 min grace period**, Feb 17th, 2017; slide your work under through the door at Maxwell-Dworkin Room 131.

Problem 1 (50 pt): Spontaneous decay vs. T_1 relaxation

(a) ^{1}H spin 1/2's are subjected to a 10-T static magnetic field in temperature 300 K. $T_1 = 5$ s (typical with ^{1}H spin 1/2's in water). Estimate the spontaneous decay time (a.k.a. life time) for a ^{1}H in the spin down state.

(b) Assume $T_1 = 1$ μs for electron spin 1/2's in a solid-state maser material in a static magnetic field of 1 T at temperature 300 K. Estimate the spontaneous decay time for an electron in the spin up state.

(c) Imagine a gas of 2-level atoms with optical resonance frequency at yellow, $T_2 = 10^{-8}$ s, and $T_1 = 10^{-5}$ s. $T = 300$ K. Estimate the spontaneous decay time for an atom in the excited state. What is the characteristic stimulated transition time due to the coupling to the background thermal (black body) radiation?

Problem 2 (200 pt): Frequency response of a 2-level "atom" — stimulated transition rate $W(\omega)$ as a function of frequency — in the rate-equation regime

^{1}H spin-1/2's are in a static magnetic field $\vec{B_0} = B_0\hat{z}$ with the resonant frequency of $\omega_0 = \gamma B_0$, spin-spin relaxation time T_2, and spin-lattice relaxation time T_1. They are excited by a transverse time-varying magnetic field along the x direction at a general frequency ω

$$\vec{B}_\perp = 2B_1\hat{x}\cos(\omega t) \tag{1}$$

where $B_1 \ll B_0$. Since we know that only the clockwise rotating field component of Eq. (1) matters (rotating wave approximation), we can just get started with

$$\vec{B}_\perp = B_1[\hat{x}\cos(\omega t) - \hat{y}\sin(\omega t)] \tag{2}$$

instead. The Rabi frequency is then given by $\omega_1 = \gamma B_1$. This problem seeks to calculate the stimulated transition rate $W(\omega)$ as a function of frequency—which represents the frequency response of the system—in the rate-equation regime ($\omega_1 T_2 \ll 1$ and $T_2 \ll T_1$). Electron spin-1/2's in many solid-state maser materials fall into the rate-equation regime but here we want to show the same physics with ^{1}H spin-1/2's so that we don't have to bother with the negative gyromagnetic ratio. We will solve the problem using the Bloch equation for the magnetization $\vec{M}$:

$$\begin{aligned} \dot{M}_x &= \gamma(M_y B_z - M_z B_y) - M_x/T_2 & (3)\\ \dot{M}_y &= \gamma(M_z B_x - M_x B_z) - M_y/T_2 & (4)\\ \dot{M}_z &= \gamma(M_x B_y - M_y B_x) + (M_0 - M_z)/T_1 & (5) \end{aligned}$$

where M_0 is the magnetization in thermal equilibrium.

(a) As $T_2 \ll T_1$ (rate-equation regime), M_z relatively slowly varies whereas for a given M_z, the amplitude of the transverse magnetization $\vec{M}_\perp = \hat{x}M_x + \hat{y}M_y$ will virtually instantaneously reach a steady-state value. We first calculate this steady-state response of $\vec{M}_\perp$ to the excitation $\vec{B}_\perp$ for a given M_z, where this response will be a function of M_z. To this end, Eq. (2) is re-written into the phasor form:

$$\tilde{\vec{B}}_\perp = \hat{x}B_1 e^{i\omega t} + \hat{y}iB_1 e^{i\omega t}. \tag{6}$$

and $\vec{M}_\perp$ can be anticipated in the same phasor form

$$\tilde{\vec{M}}_\perp = \hat{x}M_1 e^{i\omega t} + \hat{y}iM_1 e^{i\omega t}. \tag{7}$$

Here M_1 is a complex constant ($\propto$ steady-state amplitude). M_1/B_1 then represents the steady-state response for a given M_z. By using Eqs. (6) and (7) in Eqs. (3) and (4)—Eqs. (3) and (4) work for phasors as well because in each of the cross product terms of the $\vec{M}$ and $\vec{B}$ components therein, always one component is real (that is, $B_z = B_0$ and M_z are real)—, show that the transverse susceptibility linking B_1 to M_1 is given by

$$\chi(\omega) = \mu_0 \frac{M_1}{B_1} = -i\mu_0 \frac{\gamma M_z}{i(\omega - \omega_0) + 1/T_2}. \tag{8}$$

As expected, χ is a function of M_z. *Optional (not to be graded)*: can you check the Kronig-Kramers relations?

(b) Repeat the calculation of part (a) for the counter-clockwise rotating excitation field to show that the transverse susceptibility in this case is far smaller than part (a). This is the frequency-domain demonstration of the validity of the rotating wave approximation.

(c) By using the steady-state response of $\vec{M}_\perp$ to $\vec{B}_\perp$ (part (a)) in the 3rd Bloch equation [Eq. (5)], we can obtain the evolution equation for M_z. Specifically, the $\gamma(M_x B_y - M_y B_x)$ part in Eq. (5) containing only transverse components can be re-written in terms of B_1 and $\chi \propto M_z$ by using the result of (a). This requires care, however: using the phasors blindedly in $\gamma(M_x B_y - M_y B_x)$ won't work, as each term here is a product of two complex numbers. Through this calculation, show that M_z evolves according to:

$$\dot{M}_z + (M_z - M_0)/T_1 = \gamma B_1^2 \chi''(\omega)/\mu_0 \tag{9}$$

where $\chi'' \propto M_z$ is the imaginary part of χ. Show that this is of the same form as the rate equation for the population difference $\Delta N = N_+ - N_-$, and from this formal correspondence, prove that the stimulated transition rate W is given by

$$W(\omega) = \frac{\omega_1^2}{2} \left[\frac{1/T_2}{(\omega - \omega_0)^2 + (1/T_2^2)} \right] \tag{10}$$

Show that this gives $W(\omega_0) = T_2/2 \times \omega_1^2$, which we discussed in Lecture #3. Eq. (10) is a celebrated result, which holds for any 2-level atoms in the rate-equation regime whether the stimulation mechanism is magnetic (as in this case) or electric (as in the electron dipole resonance, in which case the Rabi oscillation frequency ω_1 will be proportional to the excitation electric field). Finally, while in the coherent Rabi dynamics, the 3-dB excitation bandwidth of the response was $2\omega_1$ (Lecture #1), show that in the rate-equation dynamics case treated here, the 3-dB excitation bandwidth is $2/T_2$.

(d) We can alternatively calculate $W(\omega)$ by using the result of the time-dependent perturbation theory in quantum mechanics: Fermi's golden rule or its variational form. For the stimulated transition from $|+\rangle$ to $|-\rangle$, the rate is

$$W = \frac{2\pi}{\hbar} \overline{|\langle - | H_\perp | + \rangle|^2} \rho(E_-) \tag{11}$$

where $H_\perp = -\gamma \vec{S} \cdot \vec{B}_\perp$ is the perturbing Hamiltonian and $\rho(E_-)$ is the density of states near the $|-\rangle$ state. In fact, a more suitable version given the T_2 broadening (Eq. (8)) is instead

$$W = \frac{1}{\hbar^2} \overline{|\langle - | H_\perp | + \rangle|^2} g(\omega). \tag{12}$$

Here $g(\omega)$ is what is called line shape function and is given by $g(\omega) \propto [(\omega - \omega_0)^2 + 1/T_2^2]^{-1}$ with the proportionality constant set by the normalization condition $\int_0^\infty g(f)df = 1$. Calculate the matrix element $\langle - | H_\perp | + \rangle$ (you will see that $\overline{|\langle - | H_\perp | + \rangle|^2} \propto \omega_1^2$, which bears the critical information that the stimulation emission rate is proportional to the excitation energy), and show that Eq. (12) leads to the same result as Eq. (10).