Applied Physics 216 — Assignment #4

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Due: **10:20am + 10 min grace period**, March 8th, 2017; slide your work under through the door at Maxwell-Dworkin Room 131.

Problem 1 (50 pt): Dephasing and exponential damping

Consider an electric dipole oscillator described by

$$\mu(t) = a_0 \cos(\omega_0 t + \phi(t)) \quad (1)$$

where the random process $\phi(t)$ models the dephasing that can be caused by a variety of perturbation mechanisms such as inter-atomic/molecular collisions (gas), collisions with phonons (crystalline solid), or dipole-dipole interactions. These perturbations usually have the white noise characteristic, with which $\phi(t)$ becomes a diffusion or random walk process with its statistical property given by[1]

$$\langle \phi(t) \rangle = 0 \quad (2)$$

$$\langle \phi(t_1)\phi(t_2) \rangle = 2D \min\{t_1, t_2\} \quad (3)$$

where D is the phase diffusion constant. $\phi(t)$ is also of Gaussian distribution at any given time t. Calculate $\langle \mu(t) \rangle$ to demonstrate its exponential damping and to find the relationship between T_2' and D.

Problem 2 (150 pt): Macroscopic Description of Electric Dipolar Transition

In Lecture #9, we derived the following two coupled differential equations that describe the dynamics of the macroscopic polarization $P_x(t)$ and population difference $\Delta N(t)$ for a collection of 2-level atoms of electric dipolar character subjected to an excitation electric field $E_1(t)$ along the $\hat{x}$ direction:

$$\frac{d^2 P_x}{dt^2} + \Delta\omega_0 \frac{dP_x}{dt} + \omega_0^2 P_x = R \Delta N E_1(t); \quad (4)$$

$$\frac{d\Delta N}{dt} + \frac{\Delta N - (\Delta N)_0}{T_1} = -\frac{2}{\hbar\omega_0} E_1(t) \frac{dP_x}{dt}. \quad (5)$$

Here $\delta\omega_0$ and R are given by

$$\frac{\Delta\omega_0}{2} = \frac{1}{T_2'} + \frac{1}{2T_1}; \quad (6)$$

$$R = \frac{2\pi\epsilon_0 c^3 \delta_r}{\omega_0^2} = \frac{2\omega_0 |e\langle 1|x|2\rangle|^2}{\hbar}. \quad (7)$$

Eqs. (4) and (5) are akin to the Bloch equation used to describe NMR; in fact, they are a varied form of what is known as the optical Bloch equation.

(a) In Lecture #9, we derived Eq. (4) by quantum-mechanically modifying the classical electric dipole oscillator model. In doing so, in Lecture #9, we assumed that $\langle 1|x|2\rangle = \langle 2|x|1\rangle$ for simplicity, but this assumption is neither always valid nor necessary. The always-valid statement about these matrix elements is $\langle 1|x|2\rangle = \langle 2|x|1\rangle^*$ (given the Hermiticity). Re-derive Eq. (4) under this general condition. The intention of this problem is not only to generalize the derivation, but also for you to study the structure of the calculation to quantum-mechanically modify the classical electrical dipole oscillator model.

[1] The notation, $\langle \cdot \rangle$, in this problem, signifies an ensemble average.

(b) Show from Eqs. (4) and (5) that the frequency-dependent stimulated transition rate—which is used in the description of the rate-equation regime dynamics where $\omega_1 T_2 \ll 1$ and $T_2 \ll T_1$—is given by

$$W(\omega) = \frac{\omega_1^2}{\Delta\omega_0} \times \left[1 + \left[\frac{2(\omega - \omega_0)}{\Delta\omega_0}\right]^2\right]^{-1} \tag{8}$$

(c) In Eqs. (4) and (5), turn off all the relaxation dynamics, with which we enter the perfectly coherent atom-field interaction regime. Solve for $P_x(t)$ and $\Delta N(t)$ for $E_1(t) = E_1 \cos(\omega_0 t)$ and with the initial condition of $\Delta N(0) = N$ where N is the total number of atoms per unit volume (that is, all the atoms are initially at the ground state $|1\rangle$). Verify the manifestation of Rabi oscillation. In solving this problem, you should figure out the initial condition for the polarization $P_x(t)$ and also it will be helpful to note that $P_x(t)$ will exhibit both fast and slow modulations (all of these properties of $P_x(t)$—and its initial condition—can be discovered from the quantum-mechanical model of the electrical dipole oscillator (Pages 5 and 6 of Lecture #9)). Show that $\Delta N(t)$ and the slow modulation of $P_x(t)$ are $\pi/2$ out of phase during the Rabi oscillation. Show that when ΔN decreases from N to $-N$, the excitation electric field performs work on the atomic collection, and when ΔN increases from $-N$ to N, the excitation electric field receives work from the atomic collection.