Applied Physics 216 — Assignment #5

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Due: **10:20am + 10 min grace period**, March 24th, 2017; slide your work under through the door at Maxwell-Dworkin Room 131.

Problem 1 (250 pt): Lamsey spectroscopy and passive atomic clock

(a) (10 pt) Verify briefly—considering the rotating wave approximation *a priori*—that the Hamiltonian of a 2-level atom (levels ϵ_1, ϵ_2; $\hbar\omega_0 = \epsilon_2 - \epsilon_1$) interacting with a sinusoidal field at frequency ω can be written into the following form

$$H = \begin{pmatrix} \epsilon_1 & -(b\hbar/2)e^{i\omega t} \\ -(b^*\hbar/2)e^{-i\omega t} & \epsilon_2 \end{pmatrix}, \tag{1}$$

regardless of whether the interaction is of electric or magnetic dipolar nature (for the magnetic dipolar case, we here limit the effective gyromagnetic ratio to a positive value). In the electric dipolar case with a sinusoidal electric field $E_1\cos(\omega t)\hat{x}$, $b = -e\langle 1|x|2\rangle E_1/\hbar$; in the magnetic dipolar case (*e.g.*, hyperfine transitions; spin-1/2 in a static magnetic field $B_0\hat{z}$; etc.) with a sinusoidal magnetic field $2B_1\cos(\omega t)\hat{x}$, $b = \gamma B_1$ where γ is the (effective) gyromagnetic ratio. ω_1 (Rabi frequency) $= |b|$ in either case. Lecture #11 assumed a real b, but here we will lift that assumption, as b is in general a complex number.

(b) (95 pt) Let the atom's quantum state $|\psi(t)\rangle = c_1(t)|1\rangle + c_2(t)|2\rangle$ evolve according to the Hamiltonian of Eq. (1). The state at one arbitrary time t_1 and that after an arbitrary time delay Q are connected by

$$\begin{pmatrix} c_1(t_1+Q) \\ c_2(t_1+Q) \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} c_1(t_1) \\ c_2(t_1) \end{pmatrix}. \tag{2}$$

Find the $ABCD$ matrix. As in Lecture #11, use $\Delta = \omega - \omega_0$ (often in literature Δ is defined as $\omega_0 - \omega$ but we will stick to $\Delta = \omega - \omega_0$) and $\Omega = \sqrt{\Delta^2 + |b|^2}$ for simpler expressions of the matrix elements.

(c) (95 pt)

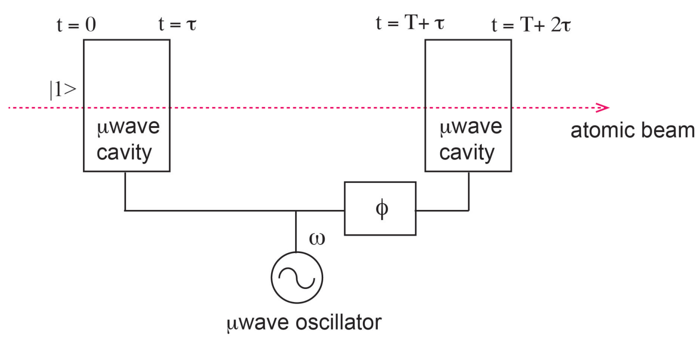

Figure 1: Ramsey spectroscopy setup.

Let ω_0 be in the microwave regime. As shown in Fig. 1, a beam consisting of the 2-level atoms passes through two separated microwave cavities, where both support sinusoidal fields at the same frequency of ω but with a general phase difference ϕ, both originating from the same microwave oscillator: that is, the signal in the first cavity is $\sim \cos(\omega t)\hat{x}$ whereas the signal in the second cavity is $\sim \cos(\omega t + \phi)\hat{x}$. The transit time in each cavity is τ. Between the two cavities, atoms fly freely ($b = 0$) over time $T \gg \tau$. Assume negligible energy and phase relaxations. By applying Eq. (2) sequentially with the $ABCD$ matrix elements properly reflecting the particular interaction in each zone, calculate the probability to find atoms at state $|2\rangle$ at the exit of the first microwave cavity (that is, $P_2(\tau) = |c_2(\tau)|^2$) and the probability to find atoms at state $|2\rangle$ at the exit of the second microwave cavity (that is, $P_2(T+2\tau) = |c_2(T+2\tau)|^2$). Plot $P_2(\tau)$ and $P_2(T+2\tau)$

as functions of the detuning Δ (for these plots, for the sake of simplicity, you can assume $\Delta^2 \gg b^2$) for $\phi = 0$ and show that the latter (Ramsey fringes) gives a far greater accuracy in determining the resonance frequency ω_0. Explain, in a couple or so sentences, how this frequency standard can be used in a passive atomic clock.

(d) (50 pt) Repeat the $P_2(T+2\tau)$ plot for $\phi = \pi$ and compare it to the $P_2(T+2\tau)$ plot for $\phi = 0$. Observe that the positions of the peaks and zeros of the Ramsey fringes differ between the two cases.[1] Explain the origin of this difference by geometrically considering the state dynamics on the Bloch sphere (page 15, Lecture #11). Such geometric representation of the quantum state evolution works generally for any 2-level atoms (that is, not only for the spin-1/2 subjected to a static magnetic field $B_0\hat{z}$ but also for electric dipolar cases), but we have not covered this generalization in class, so for this problem, you can think of spin-1/2's in a static field $B_0\hat{z}$ — *i.e.*, imagine a homogenous static magnetic field $B_0\hat{z}$ magically permeated through the entire apparatus of Fig. 1 (in practice, when hyperfine states are used for magnetic dipolar transitions, no such stringent requirement is needed).

Problem 2 (100 pt): Hanle effect

(a) Consider 4 atomic states: $|1,0,0\rangle \equiv |1\rangle$, $|2,1,0\rangle \equiv |2_0\rangle$, $|2,1,1\rangle \equiv |2_+\rangle$, and $|2,1,-1\rangle \equiv |2_-\rangle$. State $|1\rangle$ has an energy ϵ_1. States $|2_0\rangle$, $|2_+\rangle$, and $|2_-\rangle$ are degenerate with an energy ϵ_2. $\hbar\omega_0 = \epsilon_2 - \epsilon_1$. In Homework #3, you worked out the following matrix elements for the electric dipole moment operator:

$$
\begin{aligned}
-e\langle 1|x|2_+\rangle &= a, \\
-e\langle 1|x|2_-\rangle &= a, \\
-e\langle 1|y|2_+\rangle &= ia, \\
-e\langle 1|y|2_-\rangle &= -ia, \\
-e\langle 1|z|2_+\rangle &= 0, \\
-e\langle 1|z|2_-\rangle &= 0,
\end{aligned}
$$

where a is a real number which we won't bother to calculate. Show that when the atom is resonantly excited by a $\hat{y}$-polarized light, it will radiate at the same polarization.

(b) When a weak static magnetic field $B_0\hat{z}$ is applied, the degeneracy among $|2_0\rangle$, $|2_+\rangle$ and $|2_-\rangle$ is lifted (Zeeman splitting) with the energy of $|2_0\rangle$ remaining at ϵ_2 but with the energies of $|2_\pm\rangle$ being changed to $\epsilon_2 \pm \hbar\omega_L$ where $\omega_0 \gg \omega_L$. Calculate the time-dependent expected value of the electric dipole moment of the atom after it is briefly (over time τ) exposed to a $\hat{y}$-polarized light at frequency ω_0 and show that the re-emission from the atom will have both $\hat{x}$ and $\hat{y}$ polarizations.

[1] The results presented in Lecture #11 were with the assumption of $\phi = 0$. In the lecture note, I commented that the phase difference will not bring in a major difference, which is true for the frequency resolution. But the phase difference shifts the zero and peak points of the Ramsey fringes.