Applied Physics 216 — Assignment #6

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Due: **10:20am + 10 min grace period**, March 31st, 2017; slide your work under through the door at Maxwell-Dworkin Room 131.

Problem 1 (100 pt): Gain coefficient and phase shift of laser amplifier via Maxwell's equation

(a) (*Warm up*) Show the equivalence between the following two expressions for the laser gain coefficient $\gamma(f)$

$$\gamma(f) = -\Delta N \frac{\lambda^2 \delta_r}{8\pi} g(f); \tag{1}$$

$$\gamma(f) = \frac{\omega}{v}\chi''(f), \tag{2}$$

where $\chi(f)$ is the susceptibility of the 2-level electric dipolar atom system with homogenous broadening, which we calculated in Lecture #9 from the quantum-mechanically derived equation of motion, and $g(f)$ is the Lorentzian line shape function defined in Lecture #15.

(b) Imagine a monochromatic plane light wave propagating along the positive $\hat{z}$ direction in a non-magnetic dielectric laser medium (no free charge and no free current), where the population inversion $\Delta N < 0$ has been set up. The electric and magnetic fields of the light wave exhibit spatiotemporal variation according to $\sim e^{i(\omega t - kz)}$. By solving Maxwell's equations along with the constituent relation[1] $\vec{D} = \epsilon(1+\chi)\vec{E}$ in the frequency domain, derive the dispersion relation $k(\omega)$. From the dispersion relation, show that the gain coefficient is given by Eq. (2) while the phase shift per unit length is given by

$$\phi_{tot} = \frac{\omega}{v} + \frac{\omega}{2v}\chi' = \frac{\omega}{v} + \frac{f - f_0}{\Delta f_0}\gamma(f), \tag{3}$$

where f_0 is the resonance frequency of the 2-level atoms and Δf_0 is the bandwidth of $g(f)$.

Problem 2 (150 pt): Pumping, small-signal gain, and gain saturation in the 4-level laser scheme

Consider the 4-level laser scheme of Lecture #16, page 4, where the electric dipolar transition (to be used for amplification) between states $|1\rangle$ and $|2\rangle$ has a resonance frequency of f_0 falling into the optical spectral regime and a homogenous broadening characterized by the Lorentzian line shape function $g(f)$.

(a) Let pumping occur between states $|0\rangle$ and $|3\rangle$ with a stimulated transition rate W_p. No light signal (to be amplified) is applied yet. By solving the associated rate equations, show that the steady-state population difference $\Delta N_{SS,0}$ between states $|1\rangle$ and $|2\rangle$ is given by

$$\Delta N_{SS,0} = \frac{W_p(\delta_{r,21} - \delta_{r,10})}{\delta_{r,21}\delta_{r,10} + W_p\delta_{r,10} + W_p\delta_{r,21}} N. \tag{4}$$

Thus, if $\delta_{r,10} > \delta_{r,21}$, population inversion occurs. We will assume this population inversion condition from now on throughout Problem 2. Sketch $\Delta N_{SS,0}$ as a function of W_p, and express the maximum attainable $-\Delta N_{SS,0}$ in terms of $\delta_{r,21}$, $\delta_{r,10}$, and N.

(b) Now in addition to the pumping between states $|0\rangle$ and $|3\rangle$, we apply the actual light signal we seek to amplify. This light signal has a frequency f (near f_0) and an intensity $I = n_p hf$ (n_p: number of photons), and causes transitions between states $|1\rangle$ and $|2\rangle$ with a stimulated transition rate W ($\propto I$). By solving the associated rate equations, show that the new steady-state population difference ΔN_{SS} between states $|1\rangle$

[1] Here ϵ is the electric permittivity of the medium hosting the 2-level atoms.

and $|2\rangle$ and the corresponding gain coefficient $\gamma(f)$ are expressed as

$$\begin{aligned}
\Delta N_{SS} &= \frac{\Delta N_{SS,0}}{1+W/W_{sat}} = \frac{\Delta N_{SS,0}}{1+I/I_{sat}} = \frac{\Delta N_{SS,0}}{1+n_p/n_{p,sat}}; && (5)\\
\gamma(f) &= \frac{\gamma_0(f)}{1+W/W_{sat}} = \frac{\gamma_0(f)}{1+I/I_{sat}} = \frac{\gamma_0(f)}{1+n_p/n_{p,sat}}. && (6)
\end{aligned}$$

Here $\gamma_0(f)$ is the small-signal gain coefficient given by

$$\gamma_0(f) = -\Delta N_{SS,0}\frac{\lambda^2\delta_{r,21}}{8\pi}g(f) \qquad (7)$$

and W_{sat}, I_{sat}, and $n_{p,sat}$ are

$$\begin{aligned}
W_{sat} &= \frac{\delta_{r,21}\delta_{r,10}+W_p\delta_{r,10}+W_p\delta_{r,21}}{2W_p+\delta_{r,10}}; && (8)\\
I_{sat} &= \frac{\delta_{r,21}\delta_{r,10}+W_p\delta_{r,10}+W_p\delta_{r,21}}{2W_p+\delta_{r,10}} \times \frac{8\pi hf}{\delta_{r,21}\lambda^2 g(f)}; && (9)\\
n_{p,sat} &= \frac{\delta_{r,21}\delta_{r,10}+W_p\delta_{r,10}+W_p\delta_{r,21}}{2W_p+\delta_{r,10}} \times \frac{8\pi}{\delta_{r,21}\lambda^2 g(f)}. && (10)
\end{aligned}$$

(c) Observe that with increasing I, while population inversion *per se* is maintained (*i.e.*, ΔN_{SS} remains to be negative), the degree of population inversion is weakened (*i.e.*, $|\Delta N_{SS}|$ is decreased). Sketch $\gamma(f)$ as a function of I.