# Applied Physics 216 — Assignment #8

Professor: Donhee Ham
Teaching Fellow: Jundong Wu
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Due: **10:20am + 10 min grace period**, April 21st, 2017; slide your work under through the door at Maxwell-Dworkin Room 131.

**Typical characteristics of common laser transitions** — **Table useful in many parts of this set.**

| Laser medium | Transition wavelength[a] $\lambda_0$ ($\mu$m) | Stimulated transition cross section[b] $\sigma(f_0)$ (cm$^2$) | Spontaneous lifetime $1/\delta_r$ | Homogenous or inhomogenous linewidth $\Delta f_0$ or $\Delta f_i$ | Refractive index $n$ |
|---|---|---|---|---|---|
| $Ar^+$ | 0.515 | $3 \times 10^{-12}$ | 10 ns | 3.5 GHz ($\Delta f_i$) | $\sim 1$ |
| Rhodamine-6G dye | $0.56 \sim 0.64$ | $2 \times 10^{-16}$ | 5 ns | 40 THz ($\Delta f_0$, $\Delta f_i$) | $\sim 1.40$ |
| He-Ne | 0.633 | $3 \times 10^{-13}$ | 150 ns | 1.5 GHz ($\Delta f_i$) | $\sim 1$ |
| $Cr^{3+}$:$Al_2O_3$ (Ruby) | 0.694 | $2 \times 10^{-20}$ | 3 ms | 330 GHz ($\Delta f_0$) | $\sim 1.76$ |
| $Ti^{3+}$:$Al_2O_3$ (Ti:Sapphire) | $0.7 \sim 1.05$ | $3 \times 10^{-19}$ | 3.9 $\mu$s | 100 THz ($\Delta f_0$) | $\sim 1.76$ |
| $Yb^{3+}$:YAG | 1.03 | $2 \times 10^{-20}$ | 1 ms | 1 THz ($\Delta f_0$) | $\sim 1.82$ |
| $Nd^{3+}$:glass | 1.053 | $4 \times 10^{-20}$ | 370 $\mu$s | 7 THz ($\Delta f_i$) | $\sim 1.50$ |
| $Nd^{3+}$:YAG | 1.064 | $3 \times 10^{-19}$ | 230 $\mu$s | 150 GHz ($\Delta f_0$) | $\sim 1.82$ |
| InGaAsP | $1.3 \sim 1.6$ | $2 \times 10^{-16}$ | 2.5 ns | 10 THz ($\Delta f_0$) | $\sim 3.54$ |
| $Er^{3+}$: silica | 1.55 | $6 \times 10^{-21}$ | 10 ms | 5 THz ($\Delta f_0$, $\Delta f_i$) | $\sim 1.46$ |
| $CO_2$ | 10.6 | $3 \times 10^{-18}$ | 3 s | 60 MHz ($\Delta f_i$) | $\sim 1$ |

[a] The transition wavelength in the table is in reference to free space. Note, in contrast, that the wavelength used in our gain coefficient formula is in reference to the laser medium.
[b] For the definition and interpretation of stimulated transition cross section, see Problem 2.

**Problem 1 (100 pt): Doppler broadened gain coefficient**

(a) Calculate—in all necessary details—the imaginary part of the susceptibility of a Doppler broadened laser medium in a strongly inhomogeneous limit to show that the gain coefficient is given by

$$\gamma(f) = -\frac{\sqrt{\pi \ln 2}}{4\pi^2}\frac{\Delta N \lambda^2 \delta_r}{\Delta f_i}\exp\left[-4\ln 2\left(\frac{f-f_0}{\Delta f_i}\right)^2\right] \tag{1}$$

where

$$\Delta f_i = \sqrt{\frac{(8\ln 2)k_B T}{M v^2}}f_0. \tag{2}$$

(b) Compute $\Delta f_i$ for argon ion laser, He-Ne laser (neon gives the relevant atomic transition), and $CO_2$ laser at room temperature using Eq. (2), and compare these results with the measured values in Column 5 of the table above.

**Problem 2 (60 pt): Stimulated transition cross section**

Assume that a light of frequency $f$ (near $f_0$) with power $P$ is illuminated over area $A$ containing a single atom at state $|1\rangle$. The amount of power absorbed by this atom can be written as $P \times \sigma(f)/A$, where $\sigma(f)$ introduced here with the unit of area defines an "effective capture area" of the atom. This is the (stimulated transition) cross section and its value at $f = f_0$ is what is shown in the table above. The cross section $\sigma(f)$ is closely related to the gain coefficient $\gamma(f)$. To see this, assume an atomic number density $N$ in a slab of area $A$ and a thickness of $dz$; of these, $N_1$ atoms per unit volume are in state $|1\rangle$ and $N_2$ atoms per unit volume are in state $|2\rangle$. If the light of frequency $f$ and power $P$ is illuminated over the area $A$, the total effective capture area of all atoms in state $|1\rangle$ is given by $(N_1 A dz) \times \sigma(f)$. Similarly, the total effective stimulated radiation area of all atoms in state $|2\rangle$ is given by $(N_2 A dz) \times \sigma(f)$. Here we have used the fact that the effective capture area $\sigma(f)$ for a single atom in $|1\rangle$ as a receiving electric dipole antenna is the same as the effective radiation area $\sigma(f)$ for a single atom in $|2\rangle$ as a transmitting electric dipole antenna (reciprocity theorem in electromagnetism). Then the change in the light power is given by

$$dP = -\frac{(N_1 A dz) \times \sigma(f)}{A} \times P + \frac{(N_2 A dz) \times \sigma(f)}{A} \times P = -\Delta N \sigma(f) P dz. \tag{3}$$

Converting $P$ to $I$ via $P = IA$, we obtain

$$\frac{dI}{dz} = -\Delta N \sigma(f) I. \tag{4}$$

Since $I$ evolves according to $dI/dz = \gamma(f) I$, $\sigma(f)$ and $\gamma(f)$ must be directly related by

$$\sigma(f) = -\frac{\gamma(f)}{\Delta N}. \tag{5}$$

This cross section, with the unit of area per atom, offers a very useful way to represent the strength of an atomic transition in response to an applied signal.

(a) Show that in a homogeneously broadened medium $\sigma(f)$ can be written as

$$\sigma(f) = \frac{\lambda^2 \delta_r}{8\pi} g(f) \tag{6}$$

and its value at $f = f_0$ is given by

$$\sigma(f_0) = \frac{1}{4\pi^2} \times \frac{\lambda^2 \delta_r}{\Delta f_0} \tag{7}$$

Similarly, show that in a medium inhomogeneously broadened due to Doppler effect, $\sigma(f_0)$ is given by

$$\sigma(f_0) = \frac{\sqrt{\pi \ln 2}}{4\pi^2} \times \frac{\lambda^2 \delta_r}{\Delta f_i} \approx \frac{1}{8\pi} \times \frac{\lambda^2 \delta_r}{\Delta f_i} \tag{8}$$

(b) Calculate the cross section $\sigma(f_0)$ for the Ruby laser, He-Ne laser, argon ion laser, and $CO_2$ laser using the formula above together with the data in Columns 2, 4, 5, and 6 of the table of Page 1, and compare the results with the experimental cross sections in Column 3 of the same table. You will see that the match is overall reasonably good, but not exact in most cases, and it can be also quite off.

(c) A 15-cm long rod of $Nd^{3+}$:glass as a single-pass laser amplifier has a total small-signal gain (not the gain coefficient, but the actual gain) of 10 at $\lambda_0 \approx 1.053$ $\mu$m. Use the cross section data of the table of Page 1 to determine $\Delta N$ (per $cm^3$) responsible for this gain, while neglecting any medium loss.

**Problem 3 (40 pt): Gain saturation**

A homogeneously broadened single-pass laser amplifier with length $d = 5$ cm has a saturation photon flux density $n_{ph,sat}$ of $1 \times 10^{18}$ photons/($cm^2$s). When the photon flux density at the amplifier input is $2 \times 10^{15}$ photons/($cm^2$s), that at the output is $2 \times 10^{16}$ photons/($cm^2$s). Calculate the small-signal gain coefficient as well as the gain itself. What is the photon flux density at which the gain coefficient is reduced from the small-signal gain coefficient by 5 times? Determine the gain coefficient when the input photon-flux density is $4 \times 10^{19}$ photons/($cm^2$s).

**Problem 4 (370 pt): Laser oscillator exercise**

(a) (**30 pt**) Considering an argon ion laser with a resonator length of 100 cm and the loss coefficient at the half of the peak small-signal gain coefficient, estimate the number of standing wave resonance modes that can be sustained. Estimate the threshold population inversion (per $cm^3$) to acquire laser oscillation at $f_0$ when one mirror has a reflectance (local reflection coefficient squared) of 98% and the other 100% (assume no loss otherwise). What is the maximum resonator length that allows for a single-mode oscillation?

(b) (**30 pt**) A He-Ne laser has a resonator length of 30 cm and is producing a multi-mode output power of 50 mW. The peak small-signal gain coefficient is twice the loss coefficient, and the mirrors are adjusted to maximize the intensity of the strongest mode. Estimate the power of the strongest mode.

(c) (**30 pt**) A He-Ne laser has a peak small-signal gain coefficient of $\gamma_0(f_0) = 2 \times 10^{-3}$ $cm^{-1}$ and has a resonator length of 100 cm. The mirror reflectances are 100% and 97% while all other losses are negligible. Determine the number of modes that will break into self-sustained oscillation.

(d) (**30 pt**) Consider a $Yb^{3+}$:YAG laser with a $Yb^{3+}$ doping density of $1.4 \times 10^{20}$ $cm^{-3}$. First assuming thermal equilibrium at room temperature 300 K (no pumping), calculate the absorption coefficient (due to the $Yb^{3+}$ dopants) at frequency $f = f_0$. Second, assuming the mirror reflectances of 80% and 100% but neglecting any other loss, calculate the threshold population inversion (per $cm^3$) required for oscillation start up at $f = f_0$ (resonator length: 6 cm).

(e) (**30 pt**) In a homogenously broadened laser, argue that the small atomic pulling of the $l$-th standing wave resonance mode frequency $f_l = (v/2d)l$ of the pure mirror resonator into a new mode frequency $f_l'$ towards $f_0$ can be approximately quantified as

$$\frac{f_l' - f_l}{f_F} \approx \frac{\gamma(f_l)d}{\pi} \frac{f_0 - f_l}{\Delta f_0}. \tag{9}$$

Assuming $n \approx 1$, $\Delta f_0 = 1.5$ GHz, $d = 1$ m, and $\alpha_{tot} = 0.05$/m, estimate the maximum possible oscillation frequency correction $|(f_l' - f_l)/f_F|$ for the final single mode oscillation[1] (the maximum value will be attained when $|f_0 - f_l|$ becomes the largest, whose estimation is part of this problem).

(f) (**150 pt**) In this problem we consider the temporal growth of intensity $I$ in a single-mode laser oscillator. We will ignore the spatial dependence of the intensity that in general arises due to standing wave resonance formation. Show that before the gain saturation becomes appreciable, the laser oscillator's initial start-up dynamics builds the intensity $I$ according to

$$I(t) = I_0 \exp\left[\frac{(\gamma_0/\alpha_{tot}) - 1}{\tau_p} t\right], \tag{10}$$

where $\tau_p$ is the photon lifetime, $\gamma_0$ is the small-signal gain coefficient at the oscillation frequency, $\alpha_{tot}$ is the overall loss coefficient, and $I_0$ is the initial intensity (*e.g.*, due to noise). The ratio $\gamma_0/\alpha_{tot} \equiv r$ above represents how strong the initial (small-signal) gain is as compared to the loss. Now, Eq. (10) based on a fixed small-signal gain coefficient will become invalid once $I(t)$ grows sufficient and starts saturating the gain coefficient. If we seek to describe the entire evolution of $I(t)$ from initial start-up all the way to final oscillation with gain saturation, we must use the intensity-dependent (and thus time-dependent) gain coefficient[2] $\gamma(t)$,

$$\gamma(t) = \frac{\gamma_0}{1 + 2I(t)/I_{sat}}, \tag{11}$$

in solving a proper different equation for $I(t)$. Since the differential equation is difficult to tackle with Eq. (11), consider the case where $I(t) \ll I_{sat}$ for all time, even in the final oscillation.[3] In this case, Eq. (11) can be approximated into

$$\gamma(t) \approx \gamma_0 \left[1 - \frac{2I(t)}{I_{sat}}\right]. \tag{12}$$

[1]We here ignore the spatial hole burning.

[2]Can you reason why we use $2I(t)$ instead of $I(t)$ in Eq. (11)?

[3]This is the case where the small-signal gain coefficient $\gamma_0$ is not much greater than $\alpha_{tot}$ and thus the final oscillation is reached with $I(t)$ far below $I_{sat}$.

Use this intensity-dependent gain coefficient to solve the proper differential equation for $I(t)$ to obtain

$$I(t) = \frac{(I_{sat}/2)(1 - r^{-1})I_0 \exp\left[(r-1)\cdot t/\tau_p\right]}{(I_{sat}/2)(1 - r^{-1}) - I_0 + I_0 \exp\left[(r-1)\cdot t/\tau_p\right]} \tag{13}$$

where $r = \gamma_0/\alpha_{tot}$ as defined earlier. Check that during the initial start-up when the saturation effect can be ignored, Eq. (13) is reduced to Eq. (10). Calculate the final value of $I(t)$, which we will call $I_f$, and show that the corresponding final $\gamma$ value exactly equals $\alpha_{tot}$. Sketch $I(t)$ as a function of $t$ with $r = 1.1$ (and check if this is a small enough $r$ to be consistent with the approximation $I(t) \ll I_{sat}$ we are using). Show that the time it takes for the laser intensity to build from $I_0$ to 90% of $I_f$ is given by

$$\Delta t \approx \frac{\tau_p}{r-1}\ln\left(\frac{9I_f}{I_0}\right). \tag{14}$$

Estimate the above build-up time for a laser oscillator with mirror reflectances 100% and 95%, mirror distance 15 cm, no medium loss, $r = 1.1$, and $I_f/I_0 = 10^8$.

(g) (**70 pt**) Consider a single-mode laser oscillator. While one mirror is perfectly reflecting, the other has a reflectance $R < 1$, thus it exhibits a non-vanishing transmittance $T = 1 - R > 0$, which is necessary to take a portion of the laser light outside the resonator to make use of it. Let the output light's intensity be $I_{out}$. If $T$ is sufficiently small, the resonator is not appreciably leaky and the laser builds a good steady-state intensity inside the resonator, but it is difficult to take some of this outside the resonator, so $I_{out}$ will be small. On the other hand, if $T$ is sufficiently large, the resonator becomes too leaky for the laser to build up a good amount of steady-state light intensity inside the resonator, and thus $I_{out}$ will be small again. In sum, there must be an optimum $T$ that maximizes $I_{out}$. Calculate the optimum $T$ in terms of the medium loss $\alpha$ (which excludes the mirror effect), the small-signal gain coefficient $\gamma_0$, and the resonator length $d$. Feel free to approximate assuming $\alpha_{tot}d \ll 1$.